Improved control of delayed measured systems

Jens Christian Claussen and Heinz Georg Schuster
Institut für Theoretische Physik der Universität Kiel, 24098 Kiel, Germany
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In this paper, we address the question of how the control of delayed measured chaotic systems can be improved. Both unmodified Ott-Grebogi-Yorke control and difference control can be successfully applied only for a certain range of Lyapunov numbers depending on the delay time. We show that this limitation can be overcome by at least two classes of methods, namely, by rhythmic control and by the memory methods of linear predictive logging control and memory difference control.

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I. INTRODUCTION

Delay is a generic problem in the control of chaotic systems. The effective delay time τ in any feedback loop is the sum of at least three delay times, the duration of measurement, the time needed to compute the appropriate control amplitude, and the response time of the system to the applied control. The latter effect appears especially when the applied control additionally has to propagate through the system. These response times may extend to one or more cycle lengths [1].

For the formal situation of fixed-point stabilization in time-continuous control, the issue of delay has been investigated widely in control theory, dating back at least to the Smith predictor [2]. This approach mimics the, yet unknown, actual system state by a linear prediction based on the last measurement. Its time-discrete counterparts discussed in this paper allow us to place all eigenvalues of the associated linear dynamics to zero, and always ensure stability. The (time-continuous) Smith predictor with its infinite-dimensional initial condition had to be refined [3,4], giving rise to the recently active fields of *model predictive control* [5]. For fixed-point stabilization, an extension of permissible latency has been found for a modified proportional-plus-derivative controller [6].

If one wants to stabilize the dynamics of a chaotic system onto an unstable periodic orbit, one is in a special situation. In principle, a proper engineering approach could be to use the concept of sliding mode control [7], i.e., to use a comoving coordinate system and perform suitable control methods within it. However, this requires quite accurate knowledge of the whole trajectory and stable manifold, with respective numerical or experimental costs.

Therefore, direct approaches have been developed by explicitly taking into account either a Poincaré surface of section [8] or the explicit periodic orbit length [9]. This field of *controlling chaos*, or stabilization of chaotic systems, by small perturbations, in system variables [10] or control parameters [8], emerged to a widely discussed topic with applications in a broad area from technical to biological systems. Especially in fast systems [11,12] or for slow drift in parameters [13,14], difference control methods have been successful, namely, the time-continuous Pyragas scheme [9], extended time-delay autosynchronization (ETDAS) [11], and time-discrete difference control [15].

Like for the control method itself, the discussion of the measurement delay problem in chaos control has to take into account the special issues of the situation: In classical control applications, one always tries to keep the control loop latency as short as possible. In chaotic systems, however, one wants to control a fixed point of the Poincaré iteration and thus has to wait until the next crossing of the Poincaré surface of section, where the system again is in the vicinity of that fixed point.

The stability theory and the delay influence for time-continuous chaos control schemes have been studied extensively [16–20], and an improvement of control by periodic modulation has been proposed in [21]. For measurement delays that extend to a full period, however, no extension of the time-continuous Pyragas scheme is available.

In this paper, we investigate the major Poincaré-based control schemes Ott-Grebogi-Yorke (OGY) control [8] and difference feedback [15] in the presence of time delay, and focus on the question of what strategies can be used to overcome the limitations due to the delay as studied in [22]. We show how the measurement delay problem can be solved systematically for OGY control and difference control by rhythmic control and memory methods and we give constructive direct and elegant formulas for the deadbeat control in the time-discrete Poincaré iteration. While the predictive control method of linear predictive logging control (LPLC) presented below for OGY control has a direct correspondence to the Smith predictor and thus can be reviewed as its somehow straightforward implementation within the unstable subspace of the Poincaré iteration, this prediction approach does not guarantee a stable controller for difference control. However, within a class of feedback schemes linear in system parameters and system variable, there is always a unique scheme where all eigenvalues are zero, i.e., the memory difference control (MDC) scheme presented below. The method can be applied also for more than one positive Lyapunov exponent, and shows, within the validity of the linearization in the vicinity of the orbit, to be free of principal limitations in Lyapunov exponents or delay time. For zero delay (but the inherent one-period delay of difference control), MDC has been demonstrated experimentally for a chaotic electronic circuit [13] and a thermionic plasma discharge diode [14], with excellent agreement, both of stability areas and transient Lyapunov exponents, to the theory presented here.

II. CONTROL OF UNSTABLE PERIODIC ORBITS AND DELAYED MEASUREMENT

A. Ott-Grebogi-Yorke (OGY) control

The method of Ott, Grebogi, and Yorke [8] stabilizes unstable fixed points, or unstable periodic orbits utilizing a Poincaré surface of section, by feedback that is applied in the vicinity of the fixed point x^* of a discrete dynamics $x_{t+1} = f(x_t, r)$. For a chaotic flow, or corresponding experiment, the system dynamics $\dot{\vec{x}} = \vec{F}(\vec{x}, r)$ reduced to the discrete dynamics between subsequent Poincaré sections at $t_0, t_1, \ldots, t_n$. This description is fundamentally different from a stroboscopic sampling as long as the system is not on a periodic orbit, where the sequence of differences $(t_i - t_{i-1})$ would show a periodic structure.

If there is only one positive Lyapunov exponent, we can proceed considering the motion in the unstable direction only (see Appendix A), i.e., a one-dimensional iterated map. For two or more positive Lyapunov exponents, one can proceed in a similar fashion, see Appendixes B and C.

In OGY control, the control parameter r_t is made time-dependent. The amplitude of the feedback $r_t = r - r_0$ added to the control parameter r_0 is proportional by a constant ε to the distance $x - x^*$ from the fixed point, i.e., $r = r_0 + \varepsilon(x_t - x^*)$, and the feedback gain can be determined from a linearization around the fixed point, which reads, if we neglect higher-order terms,

$$\begin{aligned} f(x_t, r_o + r_t) &= f(x^*, r_0) + (x_t - x^*)\left(\frac{\partial f}{\partial x}\right)_{x^*, r_0} + r_t\left(\frac{\partial f}{\partial r}\right)_{x^*, r_0} \\ &= f(x^*, r_0) + \lambda(x_t - x^*) + \mu r_t \\ &= f(x^*, r_0) + (\lambda + \mu\varepsilon)(x_t - x^*). \end{aligned} \quad (1)$$

The second expression vanishes for $\varepsilon = -\lambda/\mu$, that is, in linear approximation the system arrives at the fixed point at the next time step, $x_{t+1} = x^*$. The uncontrolled system is assumed to be unstable in the fixed point, i.e., $|\lambda| > 1$. The system with applied control is stable if the absolute value of the eigenvalues of the iterated map is smaller than 1,

$$|x_{t+1} - x^*| = |(\lambda + \mu\varepsilon)(x_t - x^*)| < |x_t - x^*|. \quad (2)$$

Therefore, ε has to be chosen between $(-1-\lambda)/\mu$ and $(+1-\lambda)/\mu$, and this interval is of width $2/\mu$ and independent of λ, i.e., fixed points with arbitrary λ can be stabilized. This property, however, does not survive for delayed measurement: If no modification of the OGY scheme is taken into account despite a delay of τ time steps, control is delimited by a τ-dependent maximal Lyapunov number of $\lambda_{\max} = 1 + (1/\tau)$ [22]. Similarily, for difference control [15] $r_t - r_0 = \varepsilon(x_{t-\tau} - x_{t-\tau-1})$, the system is of dimension $\tau + 2$; only oscillatory repulsive fixed points can be controlled up to a Lyapunov number of

$$\lambda_{\inf} = -\left(1 + \frac{1}{\tau + \frac{1}{2}}\right)$$

[22]. Thus, delay severely reduces the number of controllable fixed points, and one has to develop special control strategies for the control of delayed measured systems.

B. Delay matching in experimental situations

Before discussing the time-discrete reduced dynamics in the Poincaré iteration, it should be clarified how this relates to an experimental control situation. At first glance, the time-discrete viewpoint seems to correspond only to a case where the delay (plus waiting time to the next Poincaré section) exactly matches the orbit length, or a multiple of it. The generic experimental situation, however, comes up with a nonmatching delay. Application of all control methods discussed here requires us to introduce an additional delay, usually by waiting for the next Poincaré crossing, so that measurement and control are applied without phase shift at the same position of the orbit. In this case, the next Poincaré crossing position x_{t+1} is a function of the values of x and r at a finite number of previous Poincaré crossings only, i.e., it does not depend on intermediate positions. Therefore, the (*a priori* infinite-dimensional) delay system reduces to a finite-dimensional iterated map.

If the delay (plus the time of the waiting mechanism to the next Poincaré crossing) is not matching the orbit length, the control schemes may perform less efficiently. Even for larger deviations from the orbit, the time between the Poincaré crossings will vary only marginally, thus a control amplitude should be available in time. In practical situations, therefore, the delay should not exceed the orbit length minus the variance of the orbit length that appears in the respective system and control setup.

In a formal sense, the Poincaré approach ensures robustness with respect to uncertainties in the orbit length, as it always ensures a synchronized reset of both trajectories and control. Between the Poincaré crossings, the control parameter is constant; the system is independent of everything *in advance of* the last Poincaré crossing. It is solely determined by the differential equation (or experimental dynamics). Thus the next crossing position is a well defined iterated function of the previous one. This is quite in contrast to the situation of a delay-differential equation (as in Pyragas control), which has an infinite-dimensional initial condition it "never gets rid of." One may proceed to stability analysis via Floquet theory [25] as investgated for continuous [16] and Poincaré-based [23,26] control schemes. Though a Poincaré crossing detection may be applied as well, the position will depend not only on the last crossing, but also on all values of the system variable within a time horizon defined by the maximum of the delay length and the (maximal) time difference between two Poincaré crossings (which are nonstroboscopic). Thus the Poincaré iteration would be a function between two infinite-dynamical spaces. Apart from further mathematical subtleties, a delay differential equation with *fixed* delay lacks the major advantage of a Poincaré map of reducing the system dynamics to a low-dimensional system.

For all control schemes discussed in this paper, the additional dimensionality is not a continuous horizon of states, but merely a finite set of values that were measured at the previous Poincaré crossings.

III. RHYTHMIC CONTROL AND STATE MEMORY CONTROL

In the remainder, we choose $r_t=0$ if no control is applied, and $x^*=0$. Before discussing the LPLC and MDC schemes in Sec. IV, which are the method of choice from a theoretical point of view, we will briefly analyze two approaches that may be favored in experimental situations, especially if an analog implementation is desired.

Rhythmic OGY control. To eliminate the additional degrees of freedom caused by the delay term, one can restrict oneself to apply control rhythmically only every $\tau+1$ time steps ($\tau+2$ for difference control), and then leave the system uncontrolled for the remaining time steps. Then $\varepsilon=\varepsilon(t)$ appears to be time-dependent with $\varepsilon(t \bmod \tau)=(\varepsilon_0,0,\ldots,0)$, and, after $(\tau+1)$ iterations we again have the same dynamics in time-delayed coordinates, but with $\lambda^{\tau+1}$ instead of λ. Equivalently,

$$x_{t+(\tau+1)} = \lambda^{\tau+1}x_t + \varepsilon_0\mu x_t, \tag{3}$$

i.e., controlling the $(\tau+1)$-fold iterate of the original system. This looks formally elegant, but leads to practically uncontrollable high effective Lyapunov numbers $\lambda^{\tau+1}$ both for large λ and large τ.

Rhythmic difference control. To enlarge the range of controllable λ, one again has the possibility to reduce the dimension of the control process in linear approximation to one by applying control every $\tau+2$ time steps,

$$x_{t+1} = \lambda x_t + \mu\varepsilon(x_{t-\tau} - x_{t-\tau-1}) = (\lambda^{\tau+1} + \mu\varepsilon\lambda - \mu\varepsilon)x_{t-\tau-1} \tag{4}$$

and the goal $x_{t+1}\overset{!}{=}0$ can be fulfilled by

$$\mu\varepsilon = -\frac{\lambda^{\tau+1}}{1-\lambda}. \tag{5}$$

One has to choose $\mu\varepsilon$ between $\mu\varepsilon_\pm=-(\lambda^{\tau+1}\pm1)/(1-\lambda)$ to achieve control as shown in Fig. 1. (The undelayed case $\tau=0$ has already been discussed in [15,23,24,27].) With rhythmic control, there is no range limit for λ, and even fixed points with positive λ can be stabilized by this method.

When using differences for periodic feedback, one still has the problem that the control gain increases by λ^τ, and noise sums up for $\tau+1$ time steps before the next control signal is applied. Additionally, now there is a singularity for $\lambda=+1$ in the "optimal" control gain given by Eq. (4), as for $\lambda\simeq+1$. Differences x_t-x_{t-1} when escaping from the fixed point are naturally small. Here one has to choose between using a large control gain (but magnifying noise and finite precision effects) or using a small control gain of order $\mu\varepsilon_-(\lambda=+1)=\tau+1$ (but having larger eigenvalues and therefore slow convergence).

State memory OGY control. If a technically simple

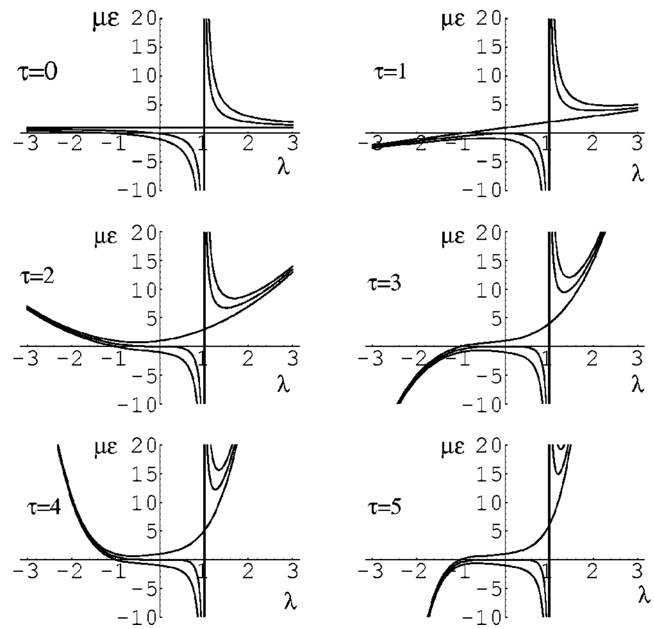

FIG. 1. Periodic difference feedback for $\tau=0,1,2,3,4,5$: Maximal, optimal, and minimal value of $\mu\varepsilon$ for given λ to obtain stabilization by control applied every $\tau+2$ time steps.

method is required, one may extend the single delay line to several delay lines, each with a gain coefficient,

$$r_t = \varepsilon_1 x_{t-1} + \varepsilon_2 x_{t-2} + \cdots + \varepsilon_{n+1}x_{t-n-1}. \tag{6}$$

For n steps memory and $\tau=1$, the control matrix is

$$\begin{pmatrix} x_{t+1} \\ \vdots \\ \\ \\ \vdots \\ x_{t-n} \end{pmatrix} = \begin{pmatrix} \lambda & \varepsilon_1 & \cdots & & & \varepsilon_n & \varepsilon_{n+1} \\ 1 & 0 & & & & & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ \vdots & & \ddots & & & & \\ & & & & \ddots & & \\ \vdots & & & & \ddots & 0 & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \vdots \\ \\ \\ \vdots \\ x_{t-n-1} \end{pmatrix}, \tag{7}$$

with the characteristic equation $(\alpha-\lambda)\alpha^{n+1}+\sum_{i=1}^{n}\varepsilon_i\alpha^{n-i}$. We can choose $\alpha_1=\alpha_2=\cdots\alpha_{n+2}=-\lambda/(n+2)$ and evaluate optimal values for all ε_i by comparing with the coefficients of the product $\Pi_{i=1}^{n+2}(\alpha-\alpha_i)$. This method allows control up to $\lambda_{\max}=2+n$, therefore arbitrary λ can be controlled if a memory length of $n>\lambda-2$ and the optimal coefficents ε_i are used.

For more than one step delay, one has the situation $\varepsilon_1=0,\ldots,\varepsilon_{\tau-1}=0$. This prohibits the "trivial pole placement" given above (choosing all α_i to the same value) and therefore reduces the maximal controllable λ and no general scheme for optimal selection of the ε_i applies. One can alternatively use the LPLC method described below, which provides an optimal control scheme.

State memory time-discrete difference control. Two other strategies that have been discussed by Socolar and Gauthier [29] are discretized versions of time-continuous methods.

Control between $\lambda=-(3+R)/(1-R)$ and $\lambda=-1$ is possible with discrete-ETDAS $(R<1)$ $r_t=\varepsilon\Sigma_{k=0}^{\infty}R^k(x_{t-k}-x_{t-k-1})$ and control between $\lambda=-(N+1)$ and $\lambda=-1$ is achieved with discrete-NTDAS (let N be a positive integer) $r_t=\varepsilon[x_t-(1/N)\Sigma_{k=0}^{N}x_{t-k}]$.

IV. IMPROVED CONTROL USING PREVIOUS CONTROL AMPLITUDES

A. Linear predictive logging control (LPLC)

We first address the OGY case where the position of the fixed point is known. If it is technically possible to store the previously applied control amplitudes $r_t, r_{t-1}, \ldots$, then one can predict the actual state x_t of the system using the linear approximation around the fixed point. That is, from the last measured value $x_{t-\tau}$ and the control amplitudes we compute estimated values iteratively by

$$y_{t-i+1} = \lambda x_{t-i} + \mu r_{t-i} \tag{8}$$

leading to a *predicted* value y_t of the actual system state. Then the original OGY formula can be applied, i.e., $r_t=-y_t\lambda/\mu$. Again, the gain parameters are linear in $x_{t-\tau}$ and all $\{r_{t'}\}$ with $t-\tau\leq t'\leq t$, and the optimal gain parameters can be expressed in terms of λ and μ.

In contrast to the state memory method presented above, the LPLC method directs the system (in linear approximation) in one time step onto the fixed point. However, when this control algorithm is switched on, one had no control applied between $t-\tau$ and $t-1$, so the trajectory has to be fairly near the orbit (in an interval with a length of order δ/λ^{τ}, where δ is the interval half-width where control is switched on). Therefore, the time one has to wait until the control can be successfully activated is of order $\lambda^{\tau-1}$ larger than in the case of undelayed control.

LPLC can also be derived as a general linear feedback in the last measured system state and all applied control amplitudes since the system was measured, and choosing the feedback gain parameters so that the linearized system has all eigenvalues zero. The linear ansatz

$$r_t = \varepsilon x_{t-\tau-i} + \eta_1 r_{t-1} + \cdots \eta_\tau r_{t-\tau} \tag{9}$$

leads to the dynamics in combined delayed coordinates

$$(x_t, x_{t-1}, \ldots, x_{t-\tau}, r_{t-1}, \ldots, r_{t-\tau})$$

$$\begin{pmatrix} x_{t+1} \\ x_t \\ \vdots \\ \vdots \\ \vdots \\ x_{t-\tau+1} \\ r_t \\ \vdots \\ \vdots \\ \vdots \\ r_{t-\tau+1} \end{pmatrix} = \begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 & \varepsilon & \eta_1 & \eta_2 & \cdots & \cdots & \eta_\tau \\ & 0 & & & & & & & & & \\ & 1 & \ddots & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \ddots & \ddots & & & & & & \\ & & & & 1 & 0 & & & & & \\ 0 & 0 & \cdots & \cdots & 0 & \varepsilon & \eta_1 & \eta_2 & \cdots & \cdots & \eta_\tau \\ & & & & & & 1 & 0 & & & \\ & & & & & & & \ddots & \ddots & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \\ \vdots \\ \vdots \\ \vdots \\ x_{t-\tau} \\ r_{t-1} \\ \vdots \\ \vdots \\ \vdots \\ r_{t-\tau} \end{pmatrix}$$

giving the characteristic polynomial

$$0 = -\alpha^{\tau}[\alpha^{\tau+1} + \alpha^{\tau}(-\lambda-\eta_1) + \alpha^{\tau-1}(\lambda\cdot\eta_1-\eta_2) + \alpha^{\tau-2}(\lambda\cdot\eta_2 - \eta_3) + \cdots + \alpha^1(\lambda\cdot\eta_{\tau-1}-\eta_\tau) + (\lambda\cdot\eta_\tau-\varepsilon)]. \tag{10}$$

All eigenvalues can be set to zero using $\varepsilon=-\lambda^{\tau+1}$ and $\eta_i=-\lambda^i$. The general formulas even for more than one positive Lyapunov exponent or multiparameter control are given in Appendix B. A straightforward extension is nonlinear predictive logging control (NLPLC) [30].

B. Memory difference control (MDC)

Now one may wish to generalize the linear predictive feedback to difference feedback, where the exact position of the fixed point may be known inaccurately. In contrast to the LPLC case, the reconstruction of the state $x_{t-\tau}$ from differences $x_{t-\tau-i}-x_{t-\tau-i-1}$ and applied control amplitudes r_{t-j} is no longer unique. As a consequence, there are infinitely many ways to compute an estimate for the present state of the system, but only a subset of these leads to a controller design ensuring convergence to the fixed point. Among these there exists a unique optimal every-step control scheme for difference feedback with minimal eigenvalues and in this sense optimal stability. This memory difference control (MDC) method has been demonstrated in an electronic experiment [13] and a plasma diode [14].

We derive the feedback rule directly from the linear ansatz,

$$r_t = \varepsilon(x_{t-\tau-i} - x_{t-\tau-i-1}) + \eta_1 r_{t-1} + \cdots \eta_\tau r_{t-\tau} \tag{11}$$

leading to the dynamics in combined delayed coordinates,

$$(x_{t+1}, x_t, \ldots, x_{t-\tau+1}, r_{t-1}, \ldots, r_{t-\tau+1})^{\mathrm{T}} = \mathbf{M} \cdot (x_t, x_{t-1}, \ldots, x_{t-\tau}, r_{t-1}, \ldots, r_{t-\tau})^{\mathrm{T}} \tag{12}$$

with

$$\mathbf{M} = \begin{pmatrix} \lambda & 0 & \cdots & 0 & \varepsilon & -\varepsilon & \eta_1 & \eta_2 - \eta_1 & \cdots & \cdots & \eta_\tau - \eta_{\tau+1} \\ 1 & 0 & & & & & & & & & \\ & 1 & \ddots & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \ddots & \ddots & & & & & & \\ & & & & 1 & 0 & & & & & \\ 0 & 0 & \cdots & 0 & \varepsilon & -\varepsilon & \eta_1 & \eta_2 - \eta_1 & \cdots & \cdots & \eta_\tau - \eta_{\tau+1} \\ & & & & & & 1 & 0 & & & \\ & & & & & & & \ddots & \ddots & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & 1 & 0 \end{pmatrix}$$

giving the characteristic polynomial

$$\begin{aligned} 0 = {} & -\alpha^\tau[\alpha^{\tau+1} + \alpha^\tau(-\lambda - \eta_1) + \alpha^{\tau-1}(\lambda \cdot \eta_1 - \eta_2) \\ & + \alpha^{\tau-2}(\lambda \cdot \eta_2 - \eta_3) \cdots + \alpha^2(\lambda \cdot \eta_{\tau-2} - \eta_{\tau-1}) \\ & + \alpha^1(\lambda \cdot \eta_{\tau-1} - \eta_\tau - \varepsilon) + (\lambda \cdot \eta_\tau + \varepsilon)]. \end{aligned} \tag{13}$$

All eigenvalues can be set to zero using $\varepsilon = -\lambda^{\tau+1}/(l-1)\,\eta_\tau = +\lambda^\tau/(l-1)$ and $\eta_i = -\lambda^i$ for $1 \leqslant i \leqslant \tau-1$. For the general case of more than one positive Lyapunov exponent or multiparameter control, see Appendix C.

V. CONCLUSIONS

We have presented methods to improve Poincaré-section-based chaos control for delayed measurement. Both for OGY control and difference control, delay delimits control, and improved control strategies have to be applied. Improved strategies contain one of the following principal ideas: rhythmic control, control with memory for previous states, or control with memory for previously applied control amplitudes.

Both rhythmic control and simple feedback control in every time step have their disadvantages: For rhythmic, control large control amplitudes, on average λ^τ/τ, are required, and noise increases by a factor $\sqrt{\tau}$. For simple feedback control, the dimension of the system is increased and the maximal controllable Lyapunov number is bounded. State memory control, however, is limited to the $\tau=1$ (OGY control) [$\tau=0$ (difference control)] case. Nevertheless, these three approaches remain valuable in situations where experimental conditions restrict the possibilities of designing the control strategy.

In general, however, the LPLC and MDC strategies proposed here allow a so-called deadbeat control with all eigenvalues zero, and they are in this sense optimal control methods. This approach has also been successfully applied in an electronic [13] and plasma [14] experiment. All parameters needed for controller design can be calculated from linearization parameters that can be fitted directly from experimental data.

APPENDIX A: TRANSFORMATION ON THE EIGENSYSTEM

Here we derive how for one unstable dimension the stabilization problem reduces to the one-dimensional case. Using a covariant basis from right eigenvectors $\vec{e}_{\mathrm{u}}$ and $\vec{e}_{\mathrm{s}}$ to eigenvectors λ_{u} und λ_{s} and the corresponding contravariant left eigenvectors $\vec{f}^{\mathrm{u}}$ and $\vec{f}^{\mathrm{s}}$ of matrix L, we can transform the linearized dynamics

$$\vec{x}_{t+1} = L\vec{x}_t + M\vec{r}_t \tag{A1}$$

with the help of $\mathbf{1} = \vec{e}_{\mathrm{u}} \cdot \cdot \vec{f}^{\mathrm{u}} + \vec{e}_{\mathrm{s}} \cdot \cdot \vec{f}^{\mathrm{s}}$ and $L = \lambda_{\mathrm{u}}\vec{e}_{\mathrm{u}} \cdot \cdot \vec{f}^{\mathrm{u}} + \lambda_{\mathrm{s}}\vec{e}_{\mathrm{s}} \cdot \cdot \vec{f}^{\mathrm{s}}$. We define as coordinates in the eigensystem $x_t^{\mathrm{u}} := \vec{f}^{\mathrm{u}}\vec{x}_t$ and $x_t^{\mathrm{s}} := \vec{f}^{\mathrm{s}}\vec{x}_t$, giving

$$x_{t+1}^{\mathrm{u}} = \lambda_{\mathrm{u}} x_t^{\mathrm{u}} + r_t \mu_{\mathrm{u}},$$

$$x_{t+1}^{\mathrm{s}} = \lambda_{\mathrm{s}} x_t^{\mathrm{s}} + r_t \mu_{\mathrm{s}}. \tag{A2}$$

We consider a general linear ansatz for the control signal,

$$r_t = \sum_{j=0}^{\infty} \vec{K}_j \cdot \vec{x}_{t-j} + \sum_{i=0}^{\infty} \eta_i \cdot r_{t-i}. \tag{A3}$$

Here $r_t = \vec{K} \cdot \vec{x}_t$ is OGY control. For r_t, it follows that

$$r_t = \sum_{j=0}^{\infty} \vec{K}_j(\vec{e}_{\rm u}\cdot\cdot\vec{f}^{\rm u} + \vec{e}_{\rm s}\cdot\cdot\vec{f}^{\rm s})\vec{x}_{t-j} + \sum_{i=0}^{\infty} \eta_i \cdot r_{t-i}$$
$$= \sum_{j=0}^{\infty} (\vec{K}_j\cdot\vec{e}_{\rm u})x^{\rm u}_{t-j} + \sum_{j=0}^{\infty} (\vec{K}_j\cdot\vec{e}_{\rm s})x^{\rm s}_{t-j} + \sum_{i=0}^{\infty} \eta_i \cdot r_{t-i}$$
$$= \sum_{j=0}^{\infty} K^{\rm u}_j x^{\rm u}_{t-j} + \sum_{j=0}^{\infty} K^{\rm s}_j x^{\rm s}_{t-j} + \sum_{i=0}^{\infty} \eta_i \cdot r_{t-i}. \quad \text{(A4)}$$

If the control in the stable direction is chosen to be zero, i.e., $\forall_j K^{\rm s}_j=0$, the dynamics decouples in two systems in the stable (unaffected in control) and in the unstable direction,

$$x^{\rm u}_{t+1} = \lambda_{\rm u} x^{\rm u}_t + \mu_{\rm u}\left(\sum_{j=0}^{\infty} K^{\rm u}_j x^{\rm u}_{t-j} + \sum_{i=0}^{\infty} \eta_i \cdot r_{t-i}\right). \quad \text{(A5)}$$

This equation is only one-dimensional, so it suffices to investigate the one-dimensional control problem if there is only one unstable direction. The generalization to higher-dimensional unstable subspaces is straightforward.

APPENDIX B: STABILIZATION OF DELAYED MAPS

We consider the motion around an unstable fixed point $\vec{x}_{t+1}=\vec{F}(\vec{x}_t,\vec{r}_t)$, where $\vec{r}_t$ and $\vec{x}_t$ have the same dimension of the phase space of the system (although it is desirable to achieve control with a minor number of control parameters). The linear time evoution around the unstable fixed point (we choose $\vec{x}^* = \vec{0}$ and $\vec{r}=\vec{0}$ in the fixed point) is given by Jacobians $D_x =: L$ and $D_r =: M$,

$$\vec{x}_{t+1} = L\vec{x}_t + M\vec{r}_t. \quad \text{(B1)}$$

In LPLC, $\vec{r}_t$ is computed from $\vec{x}_{t-\tau}$ and stored amplitudes $\vec{r}_1 \cdots \vec{r}_{t-\tau}$, being a general feedback ansatz,

$$\vec{r}_t = K\vec{x}_{t-\tau} + \sum_{j=1}^{\tau} N_j \vec{r}_{t-j}. \quad \text{(B2)}$$

Provided that det $M \neq 0$, the feedback can be chosen to

$$K = -M^{-1}L^{\tau+1}, \quad \text{(B3)}$$

$$\forall_{1\leqslant i\leqslant\tau}\, N_i = -M^{-1}L^i M. \quad \text{(B4)}$$

Iterating $\vec{x}_t = L^\tau x_{t-\tau} + \sum_{j=1}^{\tau} L^{j-1}M\vec{r}_{t-j}$, one has

$$\vec{x}_{t+1} = L\vec{x}_t + M\vec{r}_t = L^{\tau+1}x_{t-\tau} - MM^{-1}L^{\tau+1}x_{t-\tau} + \sum_{j=1}^{\tau} L^{j-1}M\vec{r}_{t-j}$$
$$- \sum_{j=1}^{\tau} MM^{-1}L^j M\vec{r}_{t-j} = \vec{0}.$$

APPENDIX C: DELAYED MEASURED MAPS—STABILIZING UNKNOWN FIXED POINTS

Now we show that even fixed points whose exact position is not given (only an approximative value is needed to determine the position of a δ-ball inside which control is switched on) can be stabilized by a difference control method that is similar to the LPLC method given above, even if one has only delayed knowledge of differences of the system variables $x_{t-\tau} - x_{t-\tau-1}$. Combined with stored values of the meantime control amplitudes $\vec{r}_1 \cdots \vec{r}_{t-\tau-1}$, we propose the control scheme

$$\vec{r}_t = K(\vec{x}_{t-\tau} - x_{t-\tau-1}) + \sum_{j=1}^{\tau+1} N_j \vec{r}_{t-j} \quad \text{(C1)}$$

with the feedback matrices

$$K = -M^{-1}L^{\tau+2}(L-\mathbf{1})^{-\mathbf{1}}, \quad \text{(C2)}$$

$$\forall_{1\leqslant i\leqslant\tau}\, N_i = -M^{-1}L^i M, \quad \text{(C3)}$$

$$N_{\tau+1} = M^{-1}L^{\tau+1}(L-\mathbf{1})^{-1}M. \quad \text{(C4)}$$

Here we have to assume that not only M but also $(L-\mathbf{1})$ is invertible. Using the linear approximation, one easily computes directly that this control leads to $\vec{x}_{t+1}=\vec{0}$, and again this is a so-called deadbeat control scheme where all eigenvalues are zero. In linearized dynamics, we have

$$\vec{x}_{t-\tau} = L\vec{x}_{t-\tau-1} + M\vec{r}_{t-\tau-1}, \quad \text{(C5)}$$

$$\vec{x}_t = L^{\tau+1}\vec{x}_{t-\tau-1} + \sum_{j=1}^{\tau+1} L^{j-1}M\vec{r}_{t-j}, \quad \text{(C6)}$$

and the next iteration reads

$$\vec{x}_{t+1} = L\vec{x}_t + M\vec{r}_t$$
$$= L^{\tau+2}\vec{x}_{t-\tau-1} + \sum_{j=1}^{\tau+1} L^j M\vec{r}_{t-j} - MM^{-1}L^{\tau+2}(L-\mathbf{1})^{-1}$$
$$\times L\vec{x}_{t-\tau-1} - MM^{-1}L^{\tau+2}(L-\mathbf{1})^{-1}M\vec{r}_{t-\tau-1}$$
$$+ MM^{-1}L^{\tau+2}(L-\mathbf{1})^{-1}\vec{x}_{t-\tau-1} - \sum_{j=1}^{\tau} MM^{-1}L^j M\vec{r}_{t-j}$$
$$+ MM^{-1}L^{\tau+1}(L-\mathbf{1})^{-1}M\vec{r}_{t-\tau-1} = \vec{0}. \quad \text{(C7)}$$

Hence delay can be overcome even for inaccurately known fixed points using a sufficient number of control parameters dim $\vec{r}$=dim $\vec{x}$, if both M and $(L-\mathbf{1})$ are invertible.

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[30] NLPLC: If it is possible to extract the first nonlinearities from the time series, control can be fundamentally improved by predicting each time step by $x_{t+1}=\lambda x_t+(\lambda_2/2)x_t^2+\mu r_t+(\mu_2/2)r_t^2+\nu x_t r_t+o(x_t^3,\ldots)$ using previously applied control amplitudes $\{r_t\}$. Then $x_{t+1}\overset{!}{=}0$ defines the optimal control method. A similar nonlinear prediction method given in [28] approximates the $x_{t+1}(x_t,r_t)$ surface directly from the time series and uses it to direct the system to any desired point. Both approaches can be regarded as a nonlinear method of model predictive control [5], applied to the Poincaré iteration dynamics. However, one has to know the fixed point x^* more accurately than in the linear case to avoid a permanent nonvanishing control amplitude and a smaller range of stability. This is crucial if the fixed point drifts in time.